Renormalized quantum dimension and Multivariable invariants for links

Cristina Ana Maria Anghel

Paris Diderot University and IMAR

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Abstract

- The aim of this talk is to present a class of multivariable link invariants constructed from a super Lie algebra of type I and their relation with Kashaev's invariants and the Volume Conjecture.

- In the first part of the talk, after a short introduction concerning the classical Reshetikhin-Turaev construction, we will describe the multivariable link invariants introduced by Geer and Patureau.

- The main idea is to use the "renormalized quantum dimension" of a module instead of the usual quantum dimension to adapt the classical Reshetikhin-Turaev method in the Lie super-algebras of type I situation.

- The second part will be devoted to the connection between the multivariable link invariants, HOMFLY-PT and Kashaev's invariants.

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Outline

1. Renormalized Reshetikhin-Turaev type construction
   - Motivation
   - Classical Reshetikhin-Turaev invariants
   - Renormalized construction

2. Geer and Patureau's Multivariable Invariants

3. Relations with other invariants
   - Renormalized invariants and Kashaev's invariants
   - ADO, Colored Jones polynomial and Kashaev's invariants
   - The Volume Conjecture

Motivation

- In 1991, Reshetikhin and Turaev defined a construction which starts with any Ribbon category and gives colored link invariants.

- They use in the definition the notion of quantum dimension of a module.

- Usually, people apply this construction for categories which come from the representation theory of some Hopf algebras (quantum groups).

- If we start with g a super-Lie algebra of type one, and we look at the quantum enveloping algebra, this is a quantum group.

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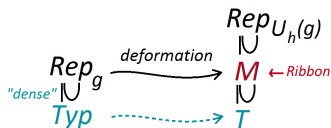
Motivation II

- We have a method to produce a Ribbon category using its representation theory.

$$
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\cup \\
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- However, if we look at the Reshetikhin-Turaev construction for M, this leads to invariants for M-colored links that vanish on any link which has at least one strand colored with a T-color.

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Definition

- Let $\mathcal{C}$ be a strict monoidal category.

- A braiding C is a natural set of isomorphisms
  $C = \{C_{V,W} \mid C_{V,W} : V \otimes W \to W \otimes V, V, W \in \mathcal{C}\}$ such that for any
  $U, V, W \in \mathcal{C}$ the following relations hold:
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- If $\mathcal{C}$ has the braiding C, a twist means a family of natural
  isomorphisms $\Theta = \{\theta_V \mid \theta_V : V \to V, V \in \mathcal{C}\}$ such that $\forall V, W \in \mathcal{C}$:
  $\theta_{V \otimes W} = C_{W,V} \circ C_{V,W}(\theta_V \otimes \theta_W)$.

- We have a duality in $\mathcal{C}$ if for any $V \in \mathcal{C}$ there is $V^* \in \mathcal{C}$ and two
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- The duality is said to be compatible with the braiding and the twist if:
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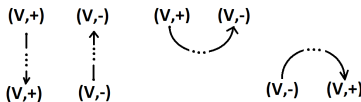# Category of framed colored tangles

Definition

Consider $\mathcal{C}$ a category. The category of $\mathcal{C}$-colored framed tangles $\mathcal{T}_{\mathcal{C}}$ is defined as follows:

$Ob(\mathcal{T}_{\mathcal{C}}) = \{(V_1, \epsilon_1), ..., (V_m, \epsilon_m) \mid m \in \mathbb{N}, \epsilon_i \in \{\pm 1\}, V_i \in \mathcal{C}\}.$
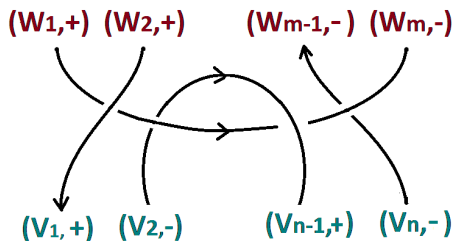
$Morph(\mathcal{T}_{\mathcal{C}})((V_1, \epsilon_1), ..., (V_m, \epsilon_m), (W_1, \delta_1), ..., (W_n, \delta_n)) =$

$$\frac{\mathcal{C}colored\ framed\ tangles\ T : (V_1, \epsilon_1), ..., (V_m, \epsilon_m) \to (W_1, \delta_1), ..., (W_n, \delta_n)}{isotopy}.$$

- *Observation* : The tangles have to respect the colors V_i.
  Once we have such a tangle, it has an induced orientation, coming from the signs ϵ_i, using the following conventions:
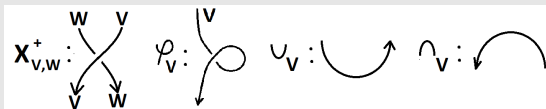
Example

Reshetikhin-Turaev functor

- Aim: Starting with any Ribbon Category $\mathcal{C}$, we'll define a functor from the category of framed $\mathcal{C}$-colored tangles to $\mathcal{C}$.

Theorem (Reshetikhin-Turaev)

Consider($\mathcal{C}$, C, Θ, b, d') a Ribbon category. Then there exists a unique functor $F_{\mathcal{C}} : \mathcal{T}_{\mathcal{C}} \to \mathcal{C}$ which is monoidal and satisfies the following local relations for any $V, W \in \mathcal{C}$:

1) $F((V, +)) = V \quad F((V, -)) = (V)^*$

2) $F(X_{V,W}^+) = C_{V,W} \quad F(\varphi_V) = \theta_V \quad F(\cup_V) = b_V \quad F(\cap_V) = d'_V$, where

Super Lie algebras of type I

Definition

A super Lie algebra is a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space $g = g_0 \oplus g_1$ with a bilinear bracket $[\,,] : g^{\otimes 2} \to g$ which satisfies:
1) $[\,x,y] = -(-1)^{\bar{x}\bar{y}}[\,y,x]$
2) Super Jacobi Identity: $[\,x,[\,y,z]\,] = [\,[\,x,y]\,,z] + (-1)^{\bar{x}\bar{y}}[\,y,[\,x,z]\,]$

- There is a splitting $g = n_- \oplus \mathfrak{H} \oplus n_+$ where h is the Cartan subalgebra of g.

- Elements of $\mathfrak{H}^*$ are called weights.

- The algebra can be described by generators and relations using a Cartan matrix.

- There are two families of super Lie algebras of type I: $sl(m,n)$ and $osp(2,2n)$.

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Reprezentation theory of g

> ### Theorem
>
> There is the following correspondence:
>
> $$\{irred.\,f.\,dimensional\,g-modules\} \longleftrightarrow highest\,weights \longleftrightarrow \Lambda = \mathbb{N}^{r-1}\times\mathbb{C}$$
>
> $$V(\lambda) \qquad\qquad\qquad\qquad\qquad \lambda \qquad ((\lambda(h_i)),\lambda(h_s))$$
>
> $$-typical$$
> $$-atypical \qquad\qquad \hookrightarrow \mathbb{N}^{r-1}\times\mathbb{Z}$$

The quantization $U_h(g)$

Definition

Let g be a super Lie algebra of type I. The quantization of g, denoted by $U_h(g)$ is the $\mathbb{C}[\,[\,h]\,]$ -super-algebra generated by three families of elements h_i, E_i and F_i, for $i \in \{1, ..., r\}$ with the relations:

$[\,h_i, h_j] = 0 \qquad [\,E_i, F_j] = \delta_{ij}\frac{q^{h_i}-q^{-h_i}}{q-q^{-1}}$

$[\,h_i, E_j] = a_{ij}E_j \qquad [\,h_i, F_j] = -a_{ij}F_j \qquad E_s^2 = F_s^2 = 0$

and quantum Serre type relations, where $[\,x, y] = xy - (-1)^{\bar{x}\bar{y}}yx$.

Definition

An $U_h(g)$-module W is called topologically free of finite rank if there is a finite dimensional g-module V with $W \simeq V[\,[\,h]\,]$ as $\mathbb{C}[\,[\,h]\,]$ -modules.

Theorem

Denote by $\mathcal{M}$=the category of topologically free of finite rank $U_h(g)$-modules. Then this is a Ribbon category.

The modified quantum dimension

- Once we obtained the Ribbon Category $\mathcal{M}$, we might think to apply the Reshetikhin-Turaev construction for that in order to obtain $\mathcal{M}-$colored link invariants. From the functoriality of F, we have that:

  - From an argument using Kontsevich integral, it follows that: $qdim(\tilde{V}(\lambda)) = 0$ for any typical color λ.

  - As a conclusion, the Reshetikhin-Turaev invariant $F(L) = 0$ for any link L colored with at least one typical color.

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Essentially, here the quantum dimension can be viewed as a function $qdim : \{weights\} \longrightarrow C[[\hbar]]$.
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- As a conclusion, the Reshetikhin-Turaev invariant $F(L) = 0$ for any link L colored with at least one typical color.

Idea

Essentially, here the quantum dimension can be viewed as a function $qdim : \{weights\} \longrightarrow C[\ [\ h]\ ]$.

The main point is to replace this quantum dimension with another function such that with a similar definition we obtain link invariants.

- In the paper "Multivariable link invariants arising from super Lie algebras of type I", N. Geer and B. Patureau defined a function $d : \{typical\ weights\} \to \mathbb{C}[\ [\ h]\ ]\ [\ h^{-1}]$ called "renormalized quantum dimension" and use this as a replacement of the quantum dimension of a module in the previous setting.

- More specifically the definition would be in the following way:

DEFINITION

Let L be a $\mathcal{M}$-colored link with at least one typical color λ. The Geer and Patureau renormalized function F' is defined as:

$$F'(L) = d(\lambda) < T_\lambda >$$

where T_λ is the tangle obtained from T by cutting the λ-colored strand.

- One point that is important about that function is the fact that it should lead to link invariants. This would mean that F' should not depend on the cutting strand colored with a typical color.

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- Let us look at the simplest example of a link, namely the Hopf link. Consider it colored with two typical colors λ, μ. We would like F' to be the same either if we cut the strand λ or μ. This is equivalent with:

$$\mathsf{d}(\lambda) \left\langle \begin{array}{c} \lambda \\ \mu \end{array} \right\rangle = \mathsf{d}(\mu) \left\langle \begin{array}{c} \mu \\ \lambda \end{array} \right\rangle$$

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$$\frac{d(\lambda)}{d(\mu)} = \frac{S'(\lambda, \mu)}{S'(\mu, \lambda)}.$$

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Proposition

Using the character formulas for g-modules, there is the following relation:

$$S'(\lambda, \mu) = \frac{\varphi_{\mu+\rho}(L'_1)}{\varphi_{\mu+\rho}(L'_0)} \cdot f(\lambda, \mu),$$

where f is a function which is symmetric in λ and μ.

- This means that the renormalized quantum dimension d should verify:

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Theorem Geer-Patureau 2010

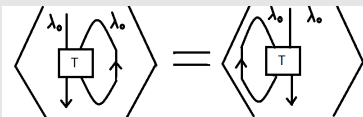Define $d : \{typical\ weights\} \to \mathbb{C}[\ [\ h\ ]\ ]\ [\ h^{-1}]$ called the renormalized quantum dimension:

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Let L be a colored link with at least one typical color λ and set $F'(L) = d(\lambda) < T_\lambda >$, where T_λ is obtained from T by cutting the λ-strand. Then F' is a well defined invariant for $\mathcal{M}$-colored links colored with at least one typical color.

- We will outline a sketch of the proof:

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There exists a special color λ_0 such that $\forall T \in \mathcal{T}((\tilde{V}(\lambda_0), \tilde{V}(\lambda_0)))$:

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As an immediate consequence of *Lemma* 1, we have:
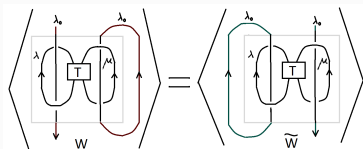
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From the monoidality of the Reshetikhin-Turaev functor, it follows that:

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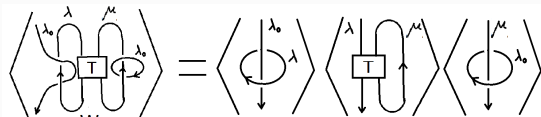
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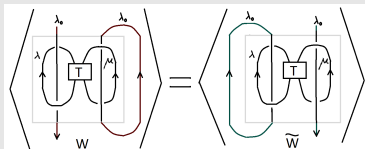
OBSERVATION

From the monoidality of the Reshetikhin-Turaev functor, it follows that:

LEMMA 3

End of the proof

Final Lemma

For any two typical weights λ and μ we have:

- This previous relation shows that F' does not depends on the cut strand so it concludes the well definition of the renormalized construction.

- We just defined invariants for links, but which have values almost in $\mathbb{C}[\,[\,h\,]\,]$. The next theorem shows that in fact they have in some sense a polynomial behavior once we fix the semicolors parametrized by $\mathbb{N}^{r-1}$ and we allow the last complex numbers to vary.

Theorem (Geer and Patureau)

Consider L a link with k components which are ordered and colored with elements $\bar{c}_i \in \mathbb{N}^{r-1}$. Denote by $\bar{c} = (\bar{c}_1, ..., \bar{c}_k)$. Then there is a Laurent polynomial in many variables $M(L, \bar{c})$ such that:
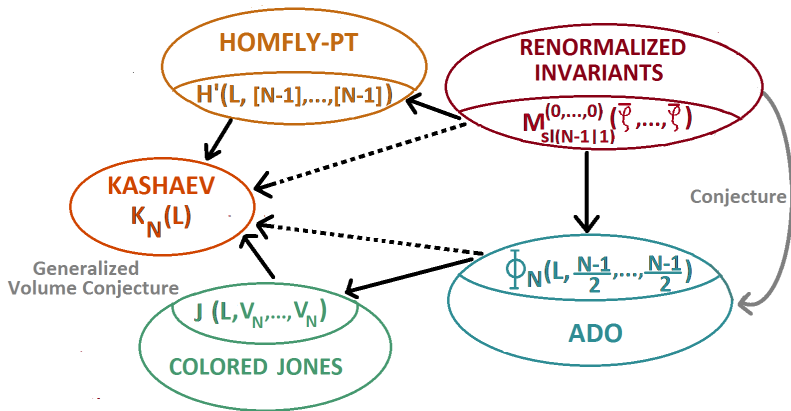
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$$M(L, \bar{c}) \in \left\{ \begin{array}{ll} M_1^{\bar{c}_1}(q, q_1)^{-1} \mathbb{Z}[\, q^{\pm 1}, q_1^{\pm 1}] & \text{if } k = 1 \\ \mathbb{Z}[\, q^{\pm 1}, q_1^{\pm 1}, ..., q_k^{\pm 1}] & \text{if } k \geq 2 \end{array} \right.$$
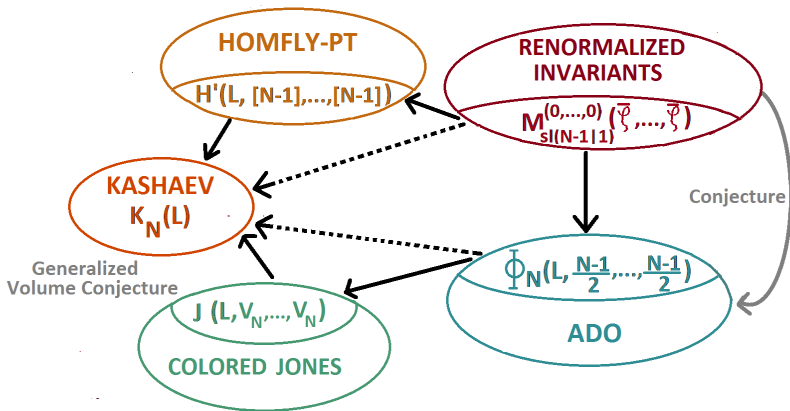
2) For any framing on L and $(\xi_1, ..., \xi_k) \in \mathbb{T}_{\bar{c}_1} \times ... \times \mathbb{T}_{\bar{c}_k}$, if we color the $i'th$ component of L with $\tilde{V}_{\xi_i}^{\bar{c}_i}$ and denote the framed colored link by $L_{\bar{\xi}}^{\bar{c}}$ then:

$$F'(L_{\bar{\xi}}^{\bar{c}}) = e^{\sum lk_{i,j} < \lambda_{\xi_i}^{\bar{c}_i}, \lambda_{\xi_j}^{\bar{c}_j} + 2\rho > \frac{h}{2}} \; M(L, \bar{c}) \mid_{q_i = e^{\frac{\xi_i h}{2}}} .$$

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RENORMALIZED INVARIANTS, HOMFLY-PT AND KHASHAEV'S INVARIANTS

- In 1995, R. Kashaev introduced a family indexed by the natural numbers of complex valued link invariants $\{K_N(L)\}_{N \in \mathbb{N}}$ using the quantum dilogarithm.

- Denote by $R = \mathbb{Q}[\, a^{\pm 1}, s^{\pm 1}, v^{\pm 1}, \frac{v - v^{-1}}{s - s^{-1}}, ([\, N - 1]\, _s!)^{-1}]$

- The HOMFLY-PT construction associates to any framed oriented link L colored with admissible Young diagrams λ an invariant $H'(L, \lambda)$ with values in R.

- For $\delta \in \mathbb{Z}^*$, consider $\psi_\delta : R \to \mathbb{Q}[\, [\, h]\,]$ by:
$$\psi_\delta(s) = q \quad \psi_\delta(v) = q^{-\delta} \quad \psi_\delta(a) = q^{-\frac{1}{\delta}}$$

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- The following theorem shows that a specialization of the renormalized multivariable invariants for the super Lie-algebra $sl(N-1 \mid 1)$ leads to Kashaev's N'th invariant:

Theorem Geer-Patureau

Let L be an oriented link. Then there is the following relation:

$$K_N(L) = N \; e^{\frac{i\pi(N-1)}{2}} \cdot M^{(0,\ldots,0)}_{sl(N-1|1)}(\xi,\ldots,\xi).$$

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If L is an oriented link and $\mu^* = (\mu_1,...,\mu_k)$ a set of admissible Young diagrams, then :

$$\psi_{m-n}(\frac{H(L,\mu^*)}{H(unknot,\mu_i)}) = \frac{F'(L,\lambda_{\mu^*})}{d(\lambda_i)}$$

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Let L be a link and L' a framed representative of L. Then there is the following relation:

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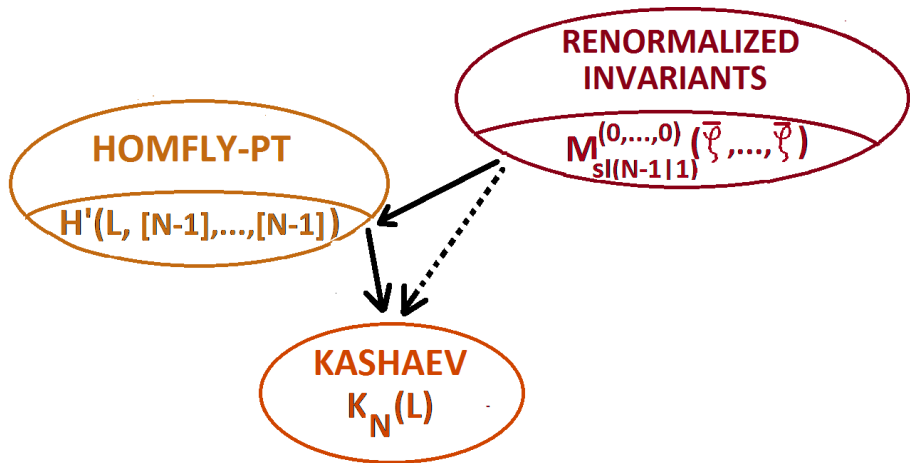If L is an oriented link and $\mu^* = (\mu_1,...,\mu_k)$ a set of admissible Young diagrams, then :

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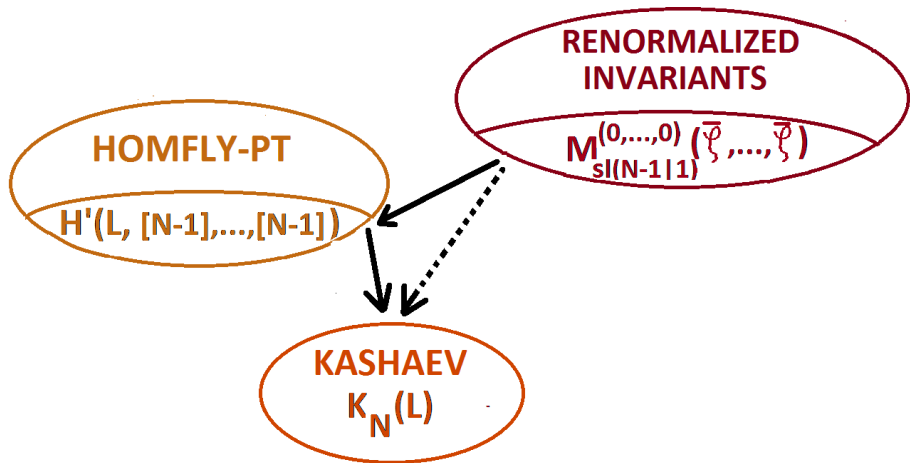$$\psi_{m-n}\left(\frac{H(L,\mu^*)}{H(unknot,\mu_i)}\right) = \frac{F'(L,\lambda_{\mu^*})}{d(\lambda_i)}$$

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ADO, Colored Jones polynomial and Kashaev's invariants

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- The colored Jones construction has as an input the Lie algebra $sl(n)$ and using the category of representations of its quantum enveloping algebra, it gives a $Rep_{U_q(sl(n))}$-colored link invariant $J(L; V_1; ...; V_k)$, $V_i \in Rep_{U_q(sl(n))}$

- In this context, H. Murakami and J. Murakami showed that there is a similar phenomenon as the one for renormalized multivariable invariants: the ADO invariants specialize to Kashaev's ones, through the colored Jones polynomial colored with $U_q(sl(2))$ representations at roots of unity.

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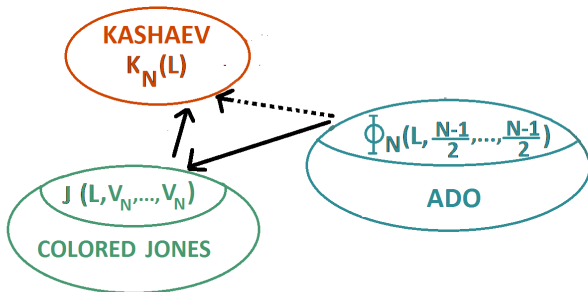
> **Theorem 1**
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> The ADO and $U_q(sl(2))$-colored Jones have specializations that coincide:
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> The $U_q(sl(2))$-colored Jones polynomials specialize to Kashaev's invariants:
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THEOREM (H. MURAKAMI AND J. MURAKAMI)

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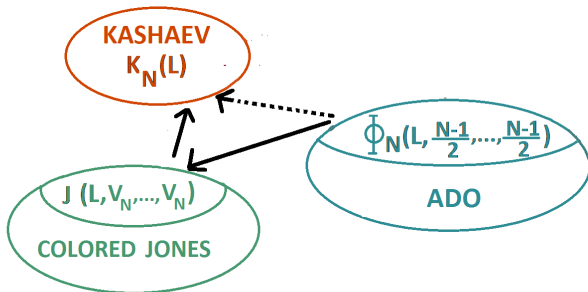The ADO and $U_q(sl(2))$-colored Jones have specializations that coincide:

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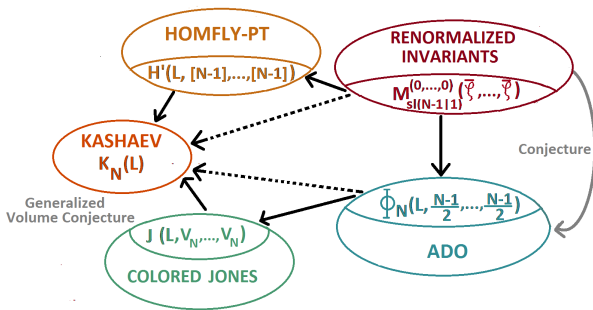
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The $U_q(sl(2))$-colored Jones polynomials specialize to Kashaev's invariants:

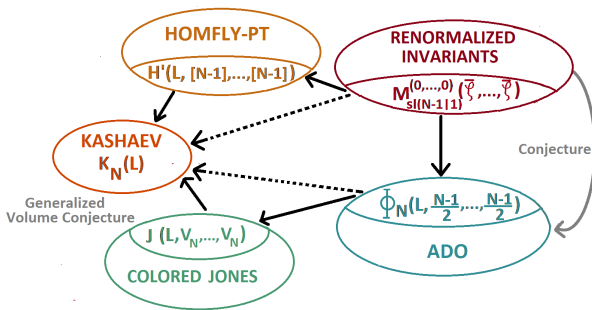$$J(L; V_N, ..., V_N)(\xi) = K_N(L)$$

COROLLARY

For any link L, the following specializations coincide:
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CONJECTURE(GEER-PATUREAU)

For any link L, and any natural colors $a_i \in \mathbb{N}$ the following relation holds:
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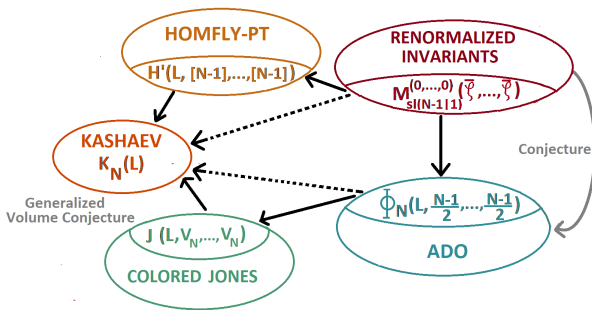
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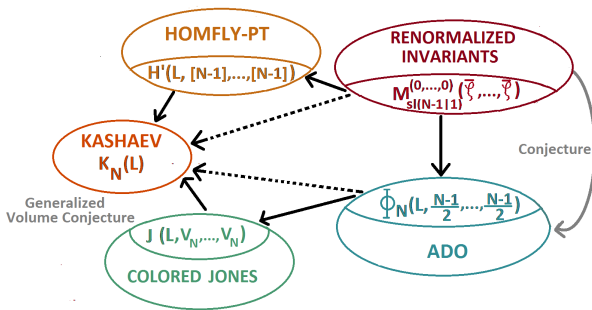$$M_{sl(N-1|1)}^{(0,...,0)}(L)(\xi, \xi^{a_1}..., \xi^{a_k}) = \phi_N(L; a_1, .., a_k)$$

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- Looking at the asymptotic behavior of his invariants, R. Khashaev proved that for some particular hyperbolic knots, the limit recovers the hyperbolic volume of the knot. He conjectured that it is true for all hyperbolic knots

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Let K be a hyperbolic knot. Then:
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The generalized volume conjecture

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Denote by v_3 the volume of the ideal tetrahedron in $\mathbb{H}^3$. For any knot K, consider the torus decomposition of the complement. The simplicial volume $\| K \|$ is the sum of the hyperbolic volumes of all hyperbolic components divided by v_3.

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The set of all finite type Vassiliev invariants detects the unknot.

- For the proof, there are the following important results:

Theorem 1

Among the knots with zero simplicial volume, the Alexander polynomial detects the unknot.

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All the coefficients of the Alexander and colored Jones polynomials for a knot are Vassiliev invariants.

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THANK YOU!